Geometry

E-OLYMP 932. The height of triangle The area of triangle is S. The length of its base is *a* greater than its height. Find the height of triangle.

► Let *h* be the height of triangle. Then its base is $h + a$. The area of the triangle is $S = ½ h (h + a)$. The values of S and *a* are given, solve the quadratic equation for *h*:

$$h^2 + ha - 2S = 0,$$

$$\text{discriminant } D = a^2 + 8S,$$

$$\text{the positive root is } h = \frac{-a + \sqrt{D}}{2}$$

E-OLYMP 1614. Angles of triangle The triangle is given. Find the value of its biggest angle.

► First, find the lengths of the sides of triangle. Then use the ***cosine theorem***

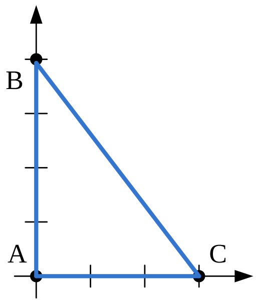
$$a^2 = b^2 + c^2 - 2 * b * c * \cos \angle BAC$$

to compute its angles. It remains to find the largest of three angles.

The distance d_{AB} between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

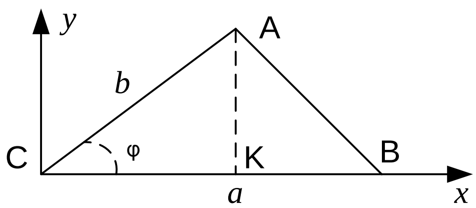
The triangle given in the problem statement has the form:



The largest angle of triangle is 90º.

E-OLYMP 5190. Construct triangle with two sides and the included angle Construct a triangle given two sides and included angle.

► Let A, B, C be three vertices of triangle, BC = *a*, AC = *b*. Place the origin of the coordinate system (0, 0) at point C, direct the abscissa axis along the vector CB. Then point B has coordinates (*a*, 0).



Draw the height AK. From tringle CAK: AK = AC sin φ, CK = AC cos φ. Considering that AC = *b*, point A will have coordinates (*b* cos φ, *b* sin φ).

Let $a(x_1, y_1)$ and $b(x_2, y_2)$ be vectors.

Absolute value of the vector a: $|a| = \sqrt{x_1^2 + y_1^2}$;

Scalar product of vectors a and b: $(a, b) = |a| * |b| * \cos(a, b) = x_1 * x_2 + y_1 * y_2$.

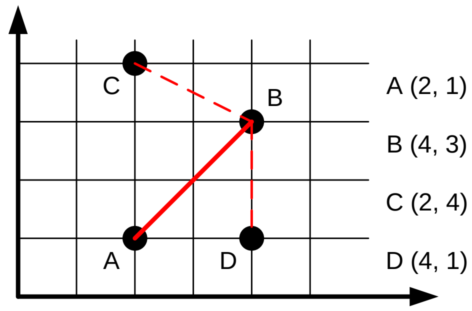
Cross (pseudoscalar) product of vectors a and b ***on a plane***:

$$a \times b = |a| * |b| * \sin(a, b) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1 = - b \times a.$$

The cross product of a and b is positive if the shortest rotation that aligns b with a is clockwise, and negative if counterclockwise. The modulus of the cross product is equal to the area of the parallelogram built on the vectors a and b.

Direction of the turn. Let there be a movement from point A to B, then from B to C. When moving, there is a

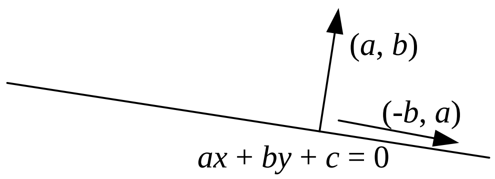
- ***left turn*** (movement occurs counterclockwise), if AB × BC > 0;
- ***right turn*** (movement occurs clockwise), if AB × BC < 0.



Vector coordinates: AB(2, 2), BC (-2, 1), BD (0, -2).

$$AB \times BC = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 2 + 4 = 6 > 0, \quad AB \times BD = \begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4 < 0$$

The equation of the line on a plane has the form $ax + by + c = 0$. The ***normal vector*** of the line has coordinates (a, b), the ***direction vector*** of the line is $(-b, a)$.

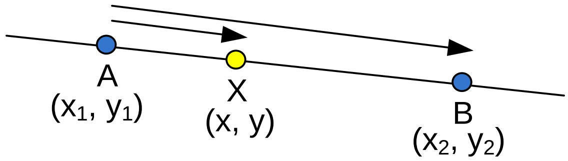


A straight line with slope k that passes through the point $(0, b)$ is given by equation

$$y = kx + b$$

The equation of the line that pass through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ has the form:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$



Proof. The line that pass through two points A and B is the locus of points X(x, y) for which the vectors AX and AB are collinear. The vectors AX($x - x_1$, $y - y_1$) and AB($x_2 - x_1$, $y_2 - y_1$) are collinear if their coordinates are proportional. The required equation of the straight line is the proportionality condition for the coordinates of the vectors AX and AB.

But in this form, horizontal and vertical lines are not representable, since they have $y_1 = y_2$ or $x_1 = x_2$. The line equation can be rewritten as:

$$(x - x_1)(y_2 - y_1) = (x_2 - x_1)(y - y_1),$$
$$x(y_2 - y_1) - x_1(y_2 - y_1) = (x_2 - x_1)y - (x_2 - x_1)y_1,$$
$$(y_2 - y_1)x - (x_2 - x_1)y - x_1y_2 + y_1x_2 = 0$$

If the equation can be written in the form $ax + by + c = 0$, then

$$a = (y_2 - y_1),$$
$$b = -(x_2 - x_1),$$
$$c = -x_1y_2 + y_1x_2$$

The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The distance from the point M(x_0, y_0) **to line** $ax + by + c = 0$ is

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

E-OLYMP **359. Circle and line** There is a circle of radius R with center at (x, y) and the line that is given with the coordinates of its two points. Find the length of the line segment inside the circle.

► The equation of the line that pass throug the points (x_1, y_1) and (x_2, y_2) has the form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1},$$

or after simplification

$$x(y_2 - y_1) + y(-x_2 + x_1) - x_1 * y_2 + y_1 * x_2 = 0$$

If the equation has the form $ax + by + c = 0$, then

$$a = y_2 - y_1,\ b = -x_2 + x_1,\ c = -x_1 * y_2 + y_1 * x_2$$

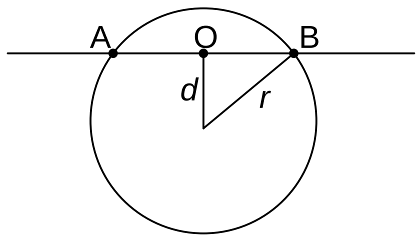
The distance from the line $ax + by + c = 0$ to the center (x_0, y_0) of the circle equals to

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

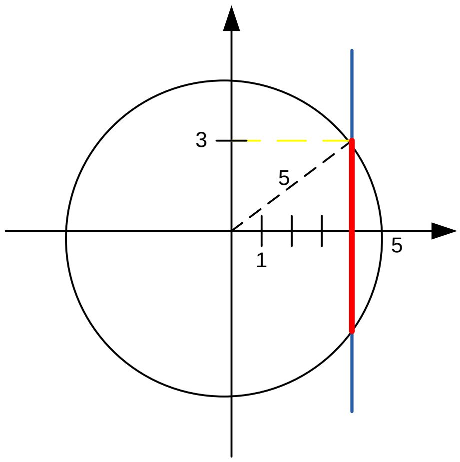
If $d > r$ then circle and line do not intersect.

If $d = r$ then circle and line touch each other.

Consider the case $d < r$ and find the length of segment AB of a line in the circle: AB = 2 * OB = 2 $\sqrt{r^2 - d^2}$.



In the given sample, a circle with radius 5 and center (0, 0) is given. A straight line has an equation $x = 4$. The length of the line's part inside the circle is 6.



Lines intersection. Let two lines $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ be given on the plane. Lines are parallel if their normal vectors are parallel, that is, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

In the case of parallelism, the lines coincide if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

If the lines are not parallel, then they intersect at one point. The point of their intersection is found by solving the system of linear equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

using the Cramer's method:

$$d = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \; d_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \; d_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \; x = \frac{d_x}{d}, \; y = \frac{d_y}{d}, \; d \neq 0$$

E-OLYMP **936. Cramer's formula** Solve the system of two linear equations with two variables using the Kramer's rule. The system of equations given in the example has the form:

$$\begin{cases} 5x_1 + 8x_2 = 11 \\ -3x_1 + 6x_2 = 15 \end{cases}$$

It is known that system has a unique solution.

► The ***determinant*** is a number equal to

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a * d - b * c$$

Consider the system of equations:

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

Multiply the first equation by b_2, and the second by b_1:

$$\begin{cases} a_1 b_2 x + b_1 b_2 y = c_1 b_2 \\ a_2 b_1 x + b_1 b_2 y = c_2 b_1 \end{cases}$$

Subtract the second equation from the first one:

$$(a_1 b_2 - a_2 b_1)\, x = c_1 b_2 - c_2 b_1$$

Or the same as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

Similarly, in the original system, multiply the first equation by a_2, and the second by a_1:

$$\begin{cases} a_1 a_2 x + a_2 b_1 y = a_2 c_1 \\ a_1 a_2 x + a_1 b_2 y = a_1 c_2 \end{cases}$$

Subtract the first equation from the second one:

$$(a_1 b_2 - a_2 b_1)\, y = a_1 c_2 - a_2 c_1$$

Or the same as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Let

$$d = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},\ d_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},\ d_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Then the solution to the system of equations can be written in the form:

$$x = \frac{d_x}{d},\ y = \frac{d_y}{d},\ d \neq 0$$

Consider the system of equations $\begin{cases} 5x_1 + 8x_2 = 11 \\ -3x_1 + 6x_2 = 15 \end{cases}$. For it we have:

$$d = \begin{vmatrix} 5 & 8 \\ -3 & 6 \end{vmatrix} = 5*6 + 3*8 = 54,$$

$$d_x = \begin{vmatrix} 11 & 8 \\ 15 & 6 \end{vmatrix} = 11*6 - 15*8 = -54,\ d_y = \begin{vmatrix} 5 & 11 \\ -3 & 15 \end{vmatrix} = 5*15 + 3*11 = 108,$$

where from

$$x = -54 / 54 = -1,\ y = 108 / 54 = 2$$

Function ***kramer*** solves a system of linear equations by Cramer's method. The roots of the system are returned in the variables x and y.

```
int kramer(double a1, double b1, double c1,
           double a2, double b2, double c2, double &x, double &y)
{
  double d = a1 * b2 - a2 * b1;
  double dx = c1 * b2 - c2 * b1;
  double dy = a1 * c2 - a2 * c1;
```

For $d = 0$, the lines are parallel.

- If $dx = 0$ (and $dy = 0$), then the lines coincide, ***return*** 2.
- If $dx \neq 0$ (in this case $dy \neq 0$ also), then the lines do not coincide (do not have common points), ***return*** 1.

```
  if (d == 0) return (dx == 0.0) + 1;
```

For $d \neq 0$, the system has a unique solution, which is calculated in the pair (x, y). In this case, we ***return*** 0.

```
  x = dx / d; y = dy / d;
  return 0;
}
```

The main part of the program. Read the input data.

```
scanf("%lf %lf %lf",&a1,&b1,&c1);
scanf("%lf %lf %lf",&a2,&b2,&c2);
```

Solve the system of equations and print the answer.

```
kramer(a1,b1,c1,a2,b2,c2,x,y);
printf("%.3lf\n%.3lf\n",x,y);
```

E-OLYMP **5134. Intersection of two lines** Find the intersection point of two lines. Each line is given with the pair of points on it.

► Construct the equations of lines passing through the pairs of given points. Then using Cramer’s rule find the intersection point of the lines. If the lines are parallel, then determine whether they coincide or not.

Three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ belong to one line if vectors $AB(x_2 - x_1, y_2 - y_1)$ and $AC(x_3 - x_1, y_3 - y_1)$ are collinear (the coordinates of AB and AC are proportional). The next equality takes place:

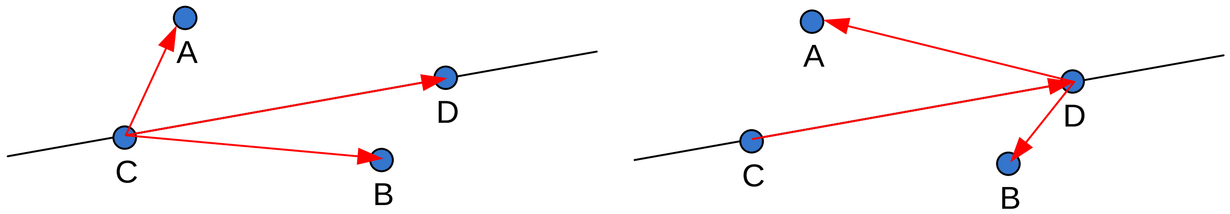
$$\frac{x_2 - x_1}{x_3 - x_1} = \frac{y_2 - y_1}{y_3 - y_1}$$

To avoid the division by zero and the possibility of using the formula for any input data, the equality should be rewritten in the form

$$(x_2 - x_1) * (y_3 - y_1) = (x_3 - x_1) * (y_2 - y_1)$$

The location of two points relative to a straight line. Two points A and B lie on the opposite sides of the line CD if the cross products CD × CA and CD × CB have different signs:

$$(CD \times CA) * (CD \times CB) < 0$$



Another condition can be written when two points A and B lie on the opposite sides of the line CD:

$$(CD \times DA) * (CD \times DB) < 0$$

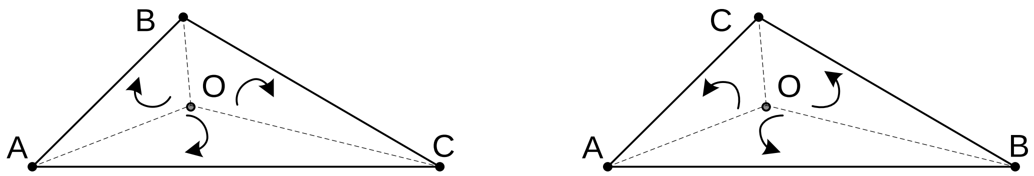
Here CDA is a left turn, CDB is a right turn.

If a line is given by equation $f(x, y) = ax + by + c = 0$, then two points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on opposite sides of it if and only if

$$f(x_1, y_1) * f(x_2, y_2) < 0$$

E-OLYMP [**666. Triangle and point**]() The triangle and the point are given on the 2D coordinate plane. Is the point In, On or Out of the triangle?

► Point O lies inside triangle ABC if and only if all three turns OAB, OBC and OCA are right (triangle ABC is oriented clockwise) or left (triangle ABC is oriented counterclockwise). That is, if all expressions OA × AB, OB × BC, OC × CA have the same sign (here × denotes the cross product).



Let $a(x_1, y_1)$ and $b(x_2, y_2)$ be the coordinates of the vectors. Then

$$a \times b = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

Point O lies on the border of the triangle if all expressions OA × AB, OB × BC, OC × CA are either non-negative or non-positive, but one of them is necessarily equal to 0. For example, if O lies on the side BC, then OB × BC = 0.

Segment equation. Let $A(x_1, y_1)$, $B(x_2, y_2)$ be the ends of segment AB. Point $X(x, y)$ belongs to segment AB if and only if | AX | + | XB | = | AB | or in coordinate form:

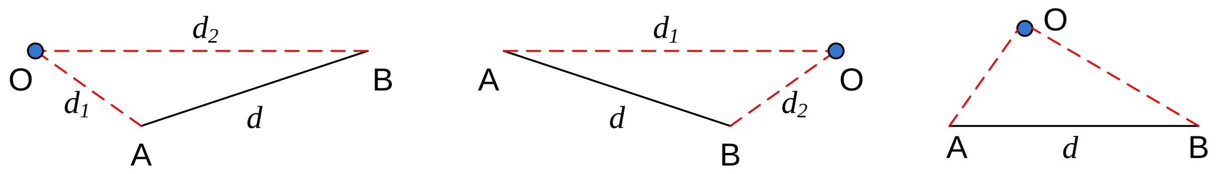
$$\sqrt{(x-x_1)^2+(y-y_1)^2} + \sqrt{(x-x_2)^2+(y-y_2)^2} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Parametric equation of a segment. Let A(x_1, y_1), B(x_2, y_2) be the ends of the segment AB. The parametric equation of the segment has the form:

$$x(t) = x_1 + (x_2 - x_1) * t,$$
$$y(t) = y_1 + (y_2 - y_1) * t,$$
$$0 \le t \le 1$$

A segment is a locus of points ($x(t)$, $y(t)$), where $0 \le t \le 1$.

The distance from the point O(x, y) ***to segment*** AB, given with its ends coordinates: A(x_1, y_1), B(x_2, y_2). Consider three cases for the relative position of a point and a segment:



Let d = AB, d_1 = OA, d_2 = OB. In the first case angle A is not acute, $d_2^2 \ge d_1^2 + d^2$, the required distance is d_1. In the second case angle B is not acute, $d_1^2 \ge d_2^2 + d^2$, the required distance is d_2. If the angles A and B are acute, then the distance from point O to segment AB equals to the length of the height of triangle OAB, that is calculated by the formula 2 * S / d, where S is the area of the triangle OAB.

Segment's point of intersection. Let A(x_1, y_1), B(x_2, y_2) are the ends of segment AB, C(x_3, y_3), D(x_4, y_4) are the ends of segment CD. Write down the parametric equations of the segments:

AB: $x(t_1) = x_1 + (x_2 - x_1) * t_1$, $y(t_1) = y_1 + (y_2 - y_1) * t_1$, $0 \le t_1 \le 1$,

CD: $x(t_2) = x_3 + (x_4 - x_3) * t_2$, $y(t_2) = y_3 + (y_4 - y_3) * t_2$, $0 \le t_2 \le 1$,

Segments intersect, if there exists such t_1, t_2 ($0 \le t_1, t_2 \le 1$), that

$$x(t_1) = x(t_2),\ y(t_1) = y(t_2)$$

Or the same as

$$x_1 + (x_2 - x_1) * t_1 = x_3 + (x_4 - x_3) * t_2$$
$$y_1 + (y_2 - y_1) * t_1 = y_3 + (y_4 - y_3) * t_2$$

We have the system of linear equations for t_1 and t_2:

$$\begin{cases} (x_2 - x_1)t_1 + (x_3 - x_4)t_2 = x_3 - x_1 \\ (y_2 - y_1)t_1 + (y_3 - y_4)t_2 = y_3 - y_1 \end{cases},$$

that can be solved using Kramer's method.

The middle point perpendicular. Let A(x_1, y_1), B(x_2, y_2) be the coordinates of the ends of segment, $ax + by + c = 0$ be the equation of the middle point perpendicular to it. Since the vectors AB($x_2 - x_1$, $y_2 - y_1$) and (a, b) are collinear, then

$$a = x_2 - x_1,\ b = y_2 - y_1$$

The middle point perpendicular passes through the point $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, the middle of segment AB, so

$$ax + by + c = (x_2 - x_1)\ \frac{x_1+x_2}{2} + (y_2 - y_1)\ \frac{y_1+y_2}{2} + c = 0,$$

where from

$$c = \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2}$$

```
void midperpend(double x1, double y1, double x2, double y2,
                double &a, double &b, double &c)
{
  a = x2 - x1;
  b = y2 - y1;
  c = (x1*x1 - x2*x2 + y1*y1 - y2*y2) / 2;
}
```

The equation of a circle centered at point (a, b) and radius r is:

$$(x - a)^2 + (y - b)^2 = r^2$$

Polar cordinates system. Each point in a polar coordinate system is defined by a radial coordinate r and a polar angle φ. The r coordinate corresponds to the distance to the pole, and the φ coordinate is the angle in the counterclockwise direction from the polar axis (the ray passing through 0 °).

The relation between polar coordinates (r, φ) and decart (x, y):

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases},\ r^2 = x^2 + y^2$$

Polar angle. Let A(x, y) be a point in decart coordinate system. To compute its polar angle φ, you can use the relations:

$$\varphi = \begin{cases} arctg\frac{y}{x}, x > 0, y \ge 0 \\ arctg\frac{y}{x} + 2\pi, x > 0, y < 0 \\ arctg\frac{y}{x} + \pi, x < 0 \\ \frac{\pi}{2}, x = 0, y > 0 \\ \frac{3\pi}{2}, x = 0, y < 0 \end{cases} \qquad \varphi = \begin{cases} arctg\frac{y}{x}, x > 0 \\ arctg\frac{y}{x} + \pi, x < 0, y \ge 0 \\ arctg\frac{y}{x} - \pi, x < 0, y < 0 \\ \frac{\pi}{2}, x = 0, y > 0 \\ -\frac{\pi}{2}, x = 0, y < 0 \end{cases}$$

$$\varphi \in [0;2\pi) \qquad \varphi \in (-\pi;\pi]$$

Compute the polar angle. The ***PolarAngle*** function computes the angle of the vector p with the abscissa axis. The angle value ranges from 0 inclusive to 2π not inclusive.

```
double PolarAngle(int x, int y)
```

```
// (x, y) != (0,0)
{
  double res = 0;
  if (x == 0)
  {
    if (y > 0)res = PI / 2; else res = 3*PI/2;
  }
  else
  {
    res = atan(1.0*y/x);
    if (x < 0) res = res + PI;
    if (res < 0) res = res + 2*PI;
  }
  return res;
}
```

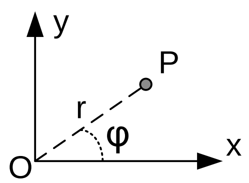
The function atan(x) computes the arctangent of x in the range from $-\pi/2$ to $\pi/2$.

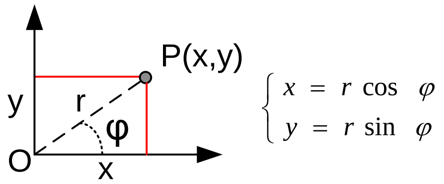
The function atan2(double y, double x) computes the arctangent of y / x between $-\pi$ and π. If $x = 0$, or both parameters are zero, function returns 0. Using ***atan2***, the ***PolarAngle*** function can be rewritten in the form:

```
double PolarAngle(int x, int y)
{
  double res = atan2(y,x);
  if (res < 0) res += 2*PI;
  return res;
}
```

E-OLYMP [2129. Polar angle of a point] Find the polar angle of the point.

► The ***polar coordinate system*** is a two-dimensional coordinate system in which each point on a plane is uniquely determined by two numbers: a polar angle φ (which is also called an azimuth or phase angle) and a polar radius r. The r coordinate is the distance from the point to the center, or pole of the coordinate system, and the φ coordinate is the angle measured counterclockwise from the ray at 0 ° (sometimes called the polar axis of the coordinate system).





Let point P have cartesian coordinates (x, y) and polar coordinates (r, φ). Then the transformation formulas from the polar coordinate system to the cartesian one have the form:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

Function atan2(double y, double x) computes the arctangent of y / x in the range $(-\pi; \pi]$. If $x = 0$, or both parameters are zero, then the function returns 0. In this problem the polar angle should be printed in the interval $[0; 2\pi)$. To do this, in the case of a negative value of ***atan2*** function, add 2π to the result.

Compute the polar angle of a point using function ***atan2***. If the result of atan2 function is negative, then 2π should be added to it, since the answer must be in the interval $[0; 2\pi)$.

```
scanf("%lf %lf",&a,&b);
res = atan2(b,a);
if (res < 0) res += 2*PI;
printf("%.6lf\n",res);
```

The area of triangle ABC, given with coordinates of its vertces A(x_1, y_1), B(x_2, y_2), C(x_3,y_3), is

$$S = \frac{1}{2} abs \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Subtract the first line from the second and third lines and compute the determinant using the third column:

$$S = \frac{1}{2} abs \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \frac{1}{2} abs \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} =$$

$$\frac{1}{2} \left| (x_2 - x_1) * (y_3 - y_1) - (x_3 - x_1) * (y_2 - y_1) \right|$$

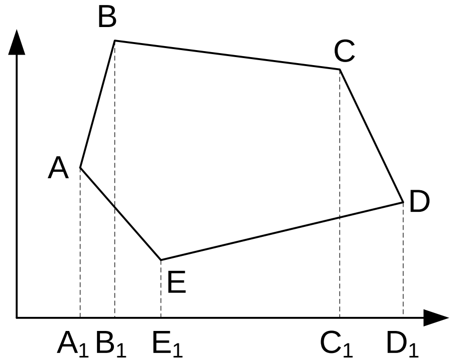
```
double TriangleArea(int x1, int y1, int x2, int y2, int x3, int y3)
{
  return abs((x2 - x1) * (y3 - y1) - (x3 - x1) * (y2 - y1)) / 2.0;
}
```

E-OLYMP **60. Area of a polygon** The coordinates of n consecutive points of polygon are given. Find the area of the polygon.

► Let $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, $A_n(x_n, y_n)$ be the coordinates of the vertices of a simple (without self-intersection) polygon, given in the order of traversing it clockwise or counterclockwise. Then its area is calculated using the trapezoidal formula:

$$S = \left| \sum_{i=1}^{n} \frac{y_{i+1} + y_i}{2} (x_{i+1} - x_i) \right|$$

where $A_{n+1}(x_{n+1}, y_{n+1}) = A_1(x_1, y_1)$. The value of the sum should be taken modulo, since it can be either positive or negative, depending on the direction of traversing the polygon.



The area of the trapezoid is positive for $x_{i+1} > x_i$ and negative for $x_i > x_{i+1}$.
The area of the polygon ABCDE equals to

$$S_{ABB_1A_1} + S_{BCC_1B_1} + S_{CDD_1C_1} - S_{DEE_1D_1} - S_{EAA_1E_1}$$

The area of a polygon can be found according to ***Surveyor*** formula:

$$S = \frac{1}{2}\left|\sum_{i=1}^{n}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}\right| = \frac{1}{2}abs\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3\end{vmatrix} + \ldots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n\end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1\end{vmatrix}\right)$$

Let O(0, 0) be the center of coordinates. Then the area of the polygon is equal to the sum of the areas of triangles OA_1A_2, OA_2A_3, OA_3A_4, … , OA_nA_1. If the triangle is oriented counterclockwise, then its area is positive. If against, then it is negative. The area of the triangle OA_iA_{i+1} is

$$\frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \\ 0 & 0 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}$$

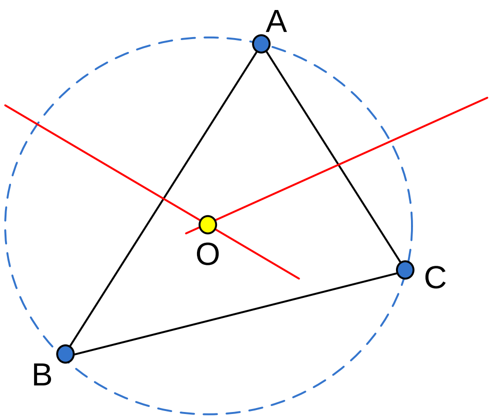
E-OLYMP **1510. Circle through three points** Three points are given on the plane that do not lie on one line. Find the equation of the circle that passes through them. Print the solution in the form

$$(x - a)^2 + (y - b)^2 = r^2 \ (1)$$

and

$$x^2 + y^2 + cx + dy + e = 0 \ (2)$$

► Construct the middle point perpendiculars to the segments AB and AC. The point of their intersection will be the center of the desired circle O. The radius of the circle is equal to the distance OA.



Define the constant EPS:

```
#define EPS 1e-6
```

Function ***kramer*** solves the system of linear equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

using the Cramer's method

$$d = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},\ d_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},\ d_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},\ x = \frac{d_x}{d},\ y = \frac{d_y}{d},\ d \neq 0$$

and returns:

- 0, if the system has a unique solution;
- 1, if the system has no solutions (lines are parallel and do not coincide);
- 2, if the system has an infinite number of solutions (lines coincide);

```
int kramer(double a1, double b1, double c1,
           double a2, double b2, double c2, double &x, double &y)
{
  double d = a1 * b2 - a2 * b1;
  double dx = c1 * b2 - c2 * b1;
  double dy = a1 * c2 - a2 * c1;

  if (!d) return (dx == 0.0) + 1;
  x = dx/d; y = dy/d;
  return 0;
}
```

Function ***midperpend*** using the coordinates of the ends of segment A(x_1, y_1) and B(x_2, y_2) constructs the midpoint perpendicular $ax + by + c = 0$. Since the vectors AB($x_2 - x_1$, $y_2 - y_1$) and (a, b) are collinear, then

$$a = x_2 - x_1,\ b = y_2 - y_1$$

The midpoint perpendicular passes through the point $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ – the middle of segment AB, so

$$ax + by + c = (x_2 - x_1)\ \frac{x_1 + x_2}{2} + (y_2 - y_1)\ \frac{y_1 + y_2}{2} + c = 0,$$

wherefrom

$$c = \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2}$$

```
void midperpend(double x1, double y1, double x2, double y2,
                double &a, double &b, double &c)
{
  a = x2 - x1;
  b = y2 - y1;
  c = (x1*x1 - x2*x2 + y1*y1 - y2*y2) / 2;
}
```

The function ***circle*** uses three points A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) to find the center of the circle (xc, yc) and the radius r that passes through them. For this, the perpendiculars to the segments AB and AC are constructed, after which the point of their intersection O is found – the center of the desired circle. The radius r is calculated as the distance between points O and A.

```
void circle(double x1, double y1, double x2, double y2, double x3,
            double y3, double &xc, double &yc, double &r)
{
```

```
  double a1,b1,c1,a2,b2,c2;
  midperpend(x1,y1,x2,y2,a1,b1,c1);
  midperpend(x1,y1,x3,y3,a2,b2,c2);
  kramer(a1,b1,-c1,a2,b2,-c2,xc,yc);
  r = sqrt((x1 - xc)*(x1 - xc) + (y1 - yc)*(y1 - yc));
}
```

The main part of the program. Read the input data.

```
while(scanf("%lf %lf %lf %lf %lf %lf",
               &x_1,&y_1,&x_2,&y_2,&x_3,&y_3) == 6)
{
```

For each test find the center (xc, yc) and the radius r of the circle that pass through three points.

```
  circle(x_1,y_1,x_2,y_2,x_3,y_3,xc,yc,r);
```

Print the equation of a circle in the form $(x – a)^2 + (y – b)^2 = r^2$.

```
  if (fabs(xc) < EPS) printf("x^2"); else
  {
    printf("(x");
    if (xc >= 0.0) printf(" - "); else printf(" + ");
    printf("%.3lf)^2",fabs(xc));
  }
  printf(" + ");
  if (fabs(yc) < EPS) printf("y^2"); else
  {
    printf("(y");
    if (yc >= 0.0) printf(" - "); else printf(" + ");
    printf("%.3lf)^2",fabs(yc));
  }
  printf(" = %.3lf^2\n",r);
```

Print the equation of a circle in the form $x^2 + y^2 + cx + dy + e = 0$.

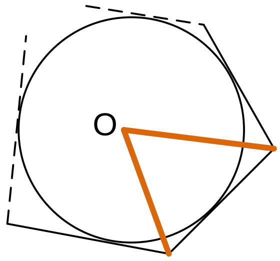
```
  printf("x^2 + y^2");

  if (fabs(xc) > EPS)
  {
    if (xc > 0.0) printf(" - "); else printf(" + ");
    printf("%.3lfx",2*fabs(xc));
  }
  if (fabs(yc) > EPS)
  {
    if (yc > 0.0) printf(" - "); else printf(" + ");
    printf("%.3lfy",2*fabs(yc));
  }
  r1 = xc*xc + yc*yc - r*r;
  if (fabs(r1) > EPS)
  {
    if (r1 > 0.0) printf(" + "); else printf(" - ");
    printf("%.3lf",fabs(r1));
  }
  printf(" = 0\n\n");
```

}

E-OLYMP **554. Inscribed circle** The circle is ***inscribed*** in a polygon, if it has a point of contact with each side of the polygon.

Determine is it possible to inscribe a circle into a given convex polygon, and if the answer is positive, find the coordinates of its center and radius.

► Construct the bisectors of two adjacent corners of the polygon and find the point (x_c, y_c) of their intersection. If a circle can be inscribed into the polygon, then this point will be its center. Compute the radius of the circle r. Find the distances from (x_c, y_c) to all sides of the polygon. If they are all equal to r, then a circle can be inscribed in the polygon. There is no other way.



Let $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ be the equations of the sides of an angle. Then the equation of the angle bisector (as a geometric set of points equidistant from the sides of the angle) has the form:

$$\frac{|a_1 x + b_1 y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2 x + b_2 y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$